A MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL
PROBLEMS WITH FUNCTIONAL DIFFERENTIAL SYSTEMS
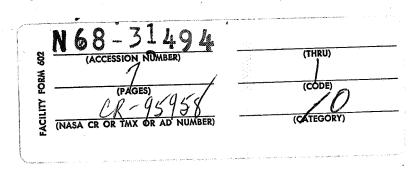
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In this note we present a maximum principle in integral form for optimal control problems with delay-differential system equations which also contain delays in the control. Recent related results for particular cases of the systems discussed below may be found in [1], [5], and [6]. Vector matrix notation will be used and we shall not distinguish between a vector and its transpose.

Let α_o and t_o be fixed in R^1 with $-\infty < \alpha_o < t_o$, $I = [\alpha_o, a)$ be a bounded interval containing $[\alpha_o, t_o]$, and put $I' = (t_o, a)$. For x continuous on I and t in I', the notation $F(x(\cdot), t)$ will mean that F is a functional in x, depending on any or all of the values $x(\tau)$, $\alpha_o \leq \tau \leq t$. $\overline{\Phi}$ will denote the class of absolutely continuous $n - 1$ vector functions on $[\alpha_o, t_o]$. Let Ω be a given convex subset of the class of all bounded Borel measurable functions u defined on I into R^r, and $\mathscr{T}$ be a given C^1 manifold in R^{2n-1}. The problem considered is that of minimizing

$$J[\overline{\varphi}, u, \overline{x}, t_1] = \int_{t_o}^{t_1} f^o(\overline{x}(\cdot), u(\cdot), t) dt$$

over $\overline{\Phi} \times \Omega \times C(I, R^{n-1}) \times I'$ subject to

(i) $\dot{\overline{x}}(t) = \overline{f}(\overline{x}(\cdot), u(\cdot), t)$ a.e. on $[t_o, t_1]$

$\overline{x}(t) = \overline{\varphi}(t)$ on $[\alpha_o, t_o]$

(ii) $(\overline{x}(t_o), \overline{x}(t_1), t_1) \in \mathscr{T}.$

We assume that $f = (f^o, \overline{f}) = (f^o, f^1, \ldots, f^{n-1})$ is an n-vector functional of the form

$$(1) \qquad f^i(\overline{x}(\cdot), u(\cdot), t) = h^i(\overline{x}(\cdot), t) + \int_{\alpha_o}^{t} u(s) d_s \eta(t, s) g^i(\overline{x}(s), t)$$

for $i = 0, 1, \ldots, n-1,$

where the integral is a Lebesgue-Stieltjes integral. Each $h^i(\overline{x}(\cdot), t)$ is assumed C^1 in $\overline{x}$ and measurable in t, and each $g^i(\overline{y}, t)$ is C^1 in $(\overline{y}, t)$ on R^n. The $r \times 1$ vector function $\eta(t, s)$ is measurable in t, s, and of bounded variation in s on $[\alpha_o, t]$. It is also assumed that the variation of η is dominated by an $L_1(I')$ function m. That is, $\bigvee_{s=\alpha_o}^{t} \eta(t, s) \leq m(t)$ for $t \in I'$. Finally, suppose that given $\overline{X}$ compact, $\overline{X} \subset R^{n-1}$, there exists an $\tilde{m}$ in $L_1(I')$ such that $h = (h^o, h^1, \ldots, h^{n-1})$ satisfies

$$|h(\overline{x}(\cdot), t)| \leq \tilde{m}(t) \qquad \cdot$$

and

$$|dh[\overline{x}(\cdot),t; \overline{\psi}]| \leq \tilde{m}(t)\|\overline{\psi}\|_t$$

for any $\overline{\psi} \in C(I,R^{n-1})$ and $\overline{x} \in C(I,\overline{X})$, where $\|\overline{\psi}\|_t = \sup\{|\overline{\psi}(s)| : s \in [\alpha_o,t]\}$ and dh is the Fréchet derivative of h with respect to $\overline{x}$. ($|A|$ denotes the Euclidean norm of A.)

If $(\overline{\varphi}^*, u^*, \overline{x}^*, t_1^*)$ is a solution of the above problem, we define the $n \times n - 1$ matrix function $\overline{\eta}^*$ for $t \in I'$, $s \in [\alpha_o,t]$ by

$$(2) \qquad \overline{\eta}^*(t,s) = \overline{\eta}_1^*(t,s) + \overline{\eta}_2^*(t,s),$$

where $\overline{\eta}_1^*$ is such that

$$dh[\overline{x}^*(\cdot),t; \overline{\psi}] = \int_{\alpha_o}^t d_s\overline{\eta}_1^*(t,s)\overline{\psi}(s)$$

for $t \in I'$ and all $\overline{\psi} \in C([\alpha_o,t],R^{n-1})$, and

$$\overline{\eta}_2^*(t,s) \equiv \begin{cases} -\int_s^t u^*(\beta)d_\beta\eta(t,\beta)g_{\overline{y}}(\overline{x}^*(\beta),t) & s < t \\ \\ 0 & s \geq t. \end{cases}$$

(The existence of $\overline{\eta}_1^*$ is guaranteed by the Riesz theorem.) Then we have the following necessary conditions:

__Theorem:__ Let $(\bar{\varphi}^*, u^*, \bar{x}^*, t_1^*)$ be a solution to the problem under the assumptions above. In addition, suppose that t_1^* is a Lebesgue point of $f(\bar{x}^*(\cdot), u^*(\cdot), t)$. Then there exists a non-trivial n-vector function $\lambda(t) = (\lambda^0(t), \bar{\lambda}(t))$ of bounded variation on $[t_0, t_1^*]$, continuous at t_1^*, satisfying:

(a) $\lambda^0(t) = \text{constant} \leq 0$, $\lambda(t_1^*) \neq 0$

$$\bar{\lambda}(t) + \int_t^{t_1^*} \lambda(\beta) \bar{\eta}^*(\beta, t) d\beta = \bar{\lambda}(t_1^*) \quad \text{for} \quad t \in [t_0, t_1^*)$$

where $\bar{\eta}^*$ is defined by (2).

(b) $\displaystyle\int_{t_0}^{t_1^*} \lambda(t) f(\bar{x}^*(\cdot), u^*(\cdot), t) dt \geq \int_{t_0}^{t_1^*} \lambda(t) f(\bar{x}^*(\cdot), u(\cdot), t) dt$

for all $u \in \Omega$.

(c) The $2n-1$ vector

$$\left(-\bar{\lambda}(t_0) + \int_{t_0}^{t_1^*} \lambda(\beta) \{ \bar{\eta}^*(\beta, \alpha_0) - \bar{\eta}^*(\beta, t_0) \} d\beta, \bar{\lambda}(t_1^*), -\lambda(t_1^*) \cdot f^*(t_1^*) \right)$$

is orthogonal to $\mathscr{F}$ at $(\bar{x}^*(t_0), \bar{x}^*(t_1^*), t_1^*)$, where $f^*(t_1^*) \equiv f(\bar{x}^*(\cdot), u^*(\cdot), t_1^*)$.

The proof of this theorem involves showing that the class of functions $\mathscr{F} = \{ F(\bar{x}(\cdot), t) : F(\bar{x}(\cdot), t) = f(\bar{x}(\cdot), u(\cdot), t), u \in \Omega \}$

is absolutely quasiconvex [2] and then using necessary conditions
for extremals given in [2]. Absolute quasiconvexity is a general-
ization of ideas due to Gamkrelidze [4], who first obtained an
integral maximum principle for control problems with ordinary dif-
ferential system equations. The inequality in (b) is a maximum
principle in integral form for the above described optimal control
problem.

In many particular cases of the systems defined by (1), one
can show that the multipliers $\overline{\lambda}$ are actually absolutely continuous
and satisfy (a) in differentiated form. This differentiated form
becomes the usual known multiplier equation for systems with simple
time lags in the state variables (see [1]). The transversality
conditions given in (c) also can be reduced to a simpler form for
many special cases of (1).

Included in (1) are many integro-differential systems and
time lag systems which appear in physical problems. For example,
if one modifies slightly the biological population model formulated
by Cooke in [3], one obtains the system equation

$$\dot{x}(t) = u(t-\tau)x(t-\tau) + \beta(t)u(t-\tau-\theta(t))x(t-\tau-\theta(t))$$

where $x(t)$ is the number in the population at time t, $u(t)$ is
the birth rate at time t, and τ is the gestation period. Systems
with

$$h(\overline{x}(\cdot),t) = \int_{\alpha_o}^{t} A(t,s)q(\overline{x}(s),t)ds,$$

which arise in the study of reactor dynamics [7], and with

$$h(\overline{x}(\cdot),t) = G(\overline{x}_t,t),$$

where $\overline{x}_t = \overline{x}(t+\theta)$, $\theta \in [-T,0]$, are also special cases of (1).

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